

# On large sequential groups

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## Abstract

We construct, using  $\diamond$ , an example of a sequential group  $G$  such that the only countable sequential subgroups of  $G$  are closed and discrete, and the only quotients of  $G$  that have a countable pseudocharacter are countable and Fréchet. We also show how to construct such a  $G$  with several additional properties (such as make  $G^2$  sequential, and arrange for every sequential subgroup of  $G$  to be closed and contain a nonmetrizable compact subspace, etc.).

Several results about  $k_\omega$  sequential groups are proved. In particular, we show that each such group is either locally compact and metrizable or contains a closed copy of the sequential fan. It is also proved that a dense proper subgroup of a non Fréchet  $k_\omega$  sequential group is not sequential extending a similar observation of T. Banach about countable  $k_\omega$  groups.

## 1 Introduction

The study of convergence in the presence of a topologically compatible algebraic structure dates back to the famous Birkhoff-Kakutani theorem on the metrizable of first-countable groups. A number of authors have studied properties such as sequentiality and Fréchetness in topological groups since then, obtaining a variety of results on metrizability of such groups, as well as discovering several pathologies exhibited by these classes of spaces.

Most of these efforts have been concentrated on studying countable sequential and Fréchet groups. One possible reason is the intuitive idea that convergence phenomena are primarily countable in nature. The Fréchet property is also inherited by arbitrary subspaces essentially reducing the study of separable Fréchet groups to that of countable ones. This research culminated in a beautiful result of Hrušák and Ramos-García establishing the independence of

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the metrizable of all countable Fréchet groups from the axioms of ZFC (their celebrated solution of the well known Malykhin's problem, see [7]).

A number of results exist on countable sequential groups, as well. The class of  $k_\omega$  countable groups (see below for the definitions of all the concepts used in this introduction) is well understood. The paper by I. Protasov and E. Zelenyuk, [12] introduced a number of techniques for the study of such groups, that have since then been applied by a host of researchers to the study of not only countable sequential but also precompact, complete, etc. group topologies (see [2], [5], and [10], for example). An elegant result by E. Zelenyuk, [16] provides a full topological classification of general countable  $k_\omega$  groups.

Not much is known about countable sequential groups that are not  $k_\omega$ , aside from some consistent examples of groups with various pathologies. In particular, there are no known ZFC examples of sequential countable groups that are not  $k_\omega$ .

Even less is known about the general, uncountable sequential groups. It is known that all compact sequential groups are metrizable (a corollary of a result by Shapirovskii, [13]). A  $\Sigma$ -product of uncountably many unit circles provides a well-known example of a Fréchet countably compact group that is not metrizable. Naturally, all separable subspaces of this group are metrizable, and thus Fréchet. It is unknown at the present time if a countably compact topological group can be made sequential and not Fréchet.

In the class of sequential groups another property comes into play: in contrast to the Fréchet property, sequentiality is not hereditary. It is an easy observation that spaces that are hereditarily sequential are Fréchet. On the other hand, every non discrete sequential space has a non discrete sequential subspace (a convergent sequence) and every separable nonmetrizable Fréchet group has a non nonmetrizable countable Fréchet subgroup.

In this paper we attempt to demonstrate that in the realm of the sequential groups 'separable' is not as easily replaced with 'countable'. We build, using  $\diamond$ , an example of an uncountable (separable if desired) sequential group  $G$  all of whose countable sequential subgroups are closed and discrete (in fact, all of its sequential subgroups are closed).

One might hope that an attempt to reduce the size of a sequential group that is not Fréchet by taking quotients might produce a 'smaller' group somehow exhibiting the non Fréchet property of the 'big' group. A rather weak condition that is desirable in a sequential group is countable pseudocharacter. Note that a sequential group can contain arbitrary sequential compact subspaces, while having a countable pseudocharacter would force all such compact subspaces to be first countable. This is indeed possible if the group is  $k_\omega$  (see Lemma 8). The example above, however, has an additional property that every quotient of countable pseudocharacter of  $G$  is countable and Fréchet.

Our example is therefore a sequential non Fréchet group that does not 'reflect' its pathologies to countable subgroups or smaller quotients. In the process of building the example, we prove a number of results about general (not necessarily countable) sequential  $k_\omega$  groups that may be of independent interest.

## 2 Definitions and terminology

We use standard set theoretic notation and terminology, see [9]. If  $X$  is a topological space and  $C \subseteq X$ ,  $x \in X$ , we write  $C \rightarrow x$  to indicate that  $C$  converges to  $x$ , i.e.  $C \subseteq^* U$  for every open  $U \ni x$ . Recall that a space  $X$  is called *sequential* if for every  $A \subseteq X$  such that  $\overline{A} \neq A$  there is a  $C \subseteq A$  such that  $C \rightarrow x \notin A$ .

Let  $A \subseteq X$ . Define the *sequential closure* of  $A$ ,  $[A]' = \{x \in X : C \rightarrow x \text{ for some } C \subseteq A\}$ . Put  $[A]_\alpha = \cup\{[A_\beta]'\} : \beta < \alpha\}$  for  $\alpha \leq \omega_1$ .

Define  $\text{so}(X) = \min\{\alpha \leq \omega_1 : [A]_\alpha = \overline{A} \text{ for every } A \subseteq X\}$ .  $\text{so}(X)$  is called the *sequential order* of  $X$ . Spaces of sequential order  $\leq 1$  are called Fréchet.

The *pseudcharacter*  $\psi(x, X)$  of  $X$  at  $x \in X$  is the smallest cardinality of a family  $\mathcal{U}$  of open neighborhoods of  $x$  such that  $\cap \mathcal{U} = \{x\}$ . When  $G$  is a topological group we write simply  $\psi(G)$ . It is well known (and easy to show) that the character (i.e. the smallest cardinality of a base of open neighborhoods at a point) and the pseudcharacter coincide for compact  $X$ .

Given an arbitrary set  $X$  and a collection of subsets  $\mathcal{K} \subseteq 2^X$ , each of which is equipped with a topology  $\tau_K$  where  $K \in \mathcal{K}$  one can introduce a topology on  $X$  as follows: a subset  $U \subseteq X$  is open in  $\tau$  if and only if each  $U \cap K$  is open in  $\tau_K$ . We will say that  $\tau$  is *determined* by  $\mathcal{K}$ . Note that as a subspace of  $X$  in this topology, the topology of  $K \in \mathcal{K}$  may be different from  $\tau_K$ . On the other hand, if each  $(K, \tau_K)$  is compact and Hausdorff and for any  $K, F \in \mathcal{K}$  the intersection  $K \cap F$  is closed in each  $K$  and  $F$  and inherits the same topology from either space then the topology of each  $K \in \mathcal{K}$  as a subspace of  $X$  is exactly  $\tau_K$ . Note that  $\tau$  is  $T_1$  if and only if each  $\tau_K$  is.

If the family  $\mathcal{K}$  above is countable, consists of compact spaces and satisfies the condition mentioned at the end of the previous paragraph,  $X$  will be called a  $k_\omega$  space.

For general groups  $1$ ,  $\cdot$ , and  $^{-1}$  stand for the unit, the group operation and the algebraic inverse, respectively. When the group is known to be abelian these change to the traditional  $0$ ,  $+$ , and  $-$ . It will be convenient to use  $\sum^k B$  for the sum  $B + \dots + B$  of  $k$  copies of some subset  $B$  of an abelian group  $G$ . Put  $\sum^0 B = B^0 = \{0\}$  (here  $B^k$  is a Cartesian product of  $B$ 's in general).

Call a family  $\mathcal{U}$  of subsets of some group  $G$  *stable* if for any  $U, V \in \mathcal{U}$  there exists a  $W \in \mathcal{U}$  such that  $WW^{-1} \subseteq U \cap V$ , and  $HU = U$  for each  $U \in \mathcal{U}$  where  $H = \cap \mathcal{U}$  is a normal subgroup of  $G$ .

If  $G$  is a topological group and  $\mathcal{U}$  is a stable family of open subsets of  $G$  it is easy to see that  $H = \cap \mathcal{U}$  is a closed subgroup of  $G$  and  $\{p(U) : U \in \mathcal{U}\}$  forms a base of open neighborhoods of  $0$  in  $G/H$  in some Hausdorff topology coarser than the quotient topology induced by the natural quotient map  $p : G \rightarrow G/H$ .

If, in addition,  $G$  is an abelian group and  $V \subseteq G$  is an open neighborhood of  $0$  then one can extend  $\mathcal{U}$  to some stable family  $\mathcal{U}'$  of open subsets of  $G$  such that there is a  $V' \in \mathcal{U}'$  with the property that  $V' \subseteq V$  and the cardinalities of  $\mathcal{U}$  and  $\mathcal{U}'$  are the same.

In the case of a non-abelian group  $G$ , the construction above has to guarantee that  $H$  is a normal subgroup of  $H$ . This can be done in the case of a separable

$G$  by ensuring that for any  $U \in \mathcal{U}$  and any  $x$  in some countable dense subset of  $G$  there exists a  $V \in \mathcal{U}$  such that  $xVx^{-1} \subseteq U$ .

The simple lemma below demonstrates the main application of stable families.

**Lemma 1.** *Let  $G$  be a topological group,  $\mathcal{U}$  be a stable family of open subsets of  $G$ ,  $E \subseteq G$  be a countable set, and  $\mathcal{K}$  be a countable family of subsets of  $G$ . If  $G$  is either abelian or separable, one can extend  $\mathcal{U}$  to a stable family  $\mathcal{U}'$  of open subsets of  $G$  such that  $|\mathcal{U}| = |\mathcal{U}'|$ ,  $p(\overline{K} \cap E) = \overline{p(K)} \cap p(E)$  for each  $K \in \mathcal{K}$ , and the restriction of  $p$  to  $E$  is one-to-one. Here  $p : G \rightarrow G/H$  is the natural quotient map and  $H = \cap \mathcal{U}'$  is a closed normal subgroup of  $G$ .*

*Proof.* Extend  $\mathcal{K}$  if necessary to guarantee that every singleton from  $E$  is in  $\mathcal{K}$ . For each pair  $(e, K) \in E \times \mathcal{K}$  such that  $e \notin \overline{K}$  find an open neighborhood of unity  $U \subseteq G$  such that  $e \cdot U \cdot U \cap \overline{K} \cdot U \cdot U = \emptyset$ . Using the remarks preceding the statement of the lemma, extend  $\mathcal{U}$  to a stable family  $\mathcal{U}'$  such that for each open set  $U$  constructed above there is a  $V \in \mathcal{U}'$  such that  $V \subseteq U$ .  $\square$

Note that extending a stable family  $\mathcal{U}$  to a stable family  $\mathcal{U}'$  of open subsets of  $G$  produces a pair of open group homomorphisms  $G \xrightarrow{p} G/H' \xrightarrow{p'} G/H$ . Here  $H = \cap \mathcal{U}$ ,  $H' = \cap \mathcal{U}'$ .

The next lemma is a corollary of a more general result.

**Lemma 2** (see [8]). *An image of a Fréchet space under an open map is Fréchet. In general, open maps do not raise the sequential order of a space.*

An uncountable group  $G$  will be called *co-countable* if every quotient of  $G$  that has countable pseudocharacter is countable.

Recall that a (necessarily abelian) group  $G$  is called *boolean* if  $a + a = 0$  for every  $a \in G$ . Every boolean group can be naturally viewed as a vector space over the two element field  $\mathbb{F}_2$ .

### 3 Test Spaces

The study of sequential spaces often makes use of a wide variety of smaller (usually countable) canonical spaces to investigate various convergence phenomena. Below we present the definitions of the test spaces used in this paper.

Let  $S_n = [\omega]^{\leq n}$ . Put  $U \subseteq S_n$  open if and only if for every  $s \in U$  the set  $\{s \frown n \in S_n : s \frown n \notin U\}$  is finite. Now each  $S_n$  is sequential and  $\text{so}(S_n) = n$  for  $n < \omega$ , whereas  $\text{so}(S_\omega) = \omega_1$ .  $S_2$  is known also as *Arens' space* while  $S_\omega$  is referred to as *Arkhangel'skii-Franklin space*. The *sequential fan*  $S(\omega)$  is defined as  $\omega^2 \cup \{0\}$  with the topology in which each point of  $\omega^2$  is isolated and the base of neighborhoods of 0 is formed by  $U_f = \{0\} \cup \{(i, j) : j \geq f(i)\}$  where  $f : \omega \rightarrow \omega$ .

It is convenient to have a 'functional' description of a closed embedding of  $S_2$  into a space which is provided by the following definition. An injective map  $s : \omega^2 \rightarrow X$  is called a *free  $S_2$ -embedding* into  $X$  (with respect to some topology

$\tau$  on  $X$ ) if  $s(i, j) \rightarrow a_i$  and  $a_i \rightarrow v$  where all  $s(i, j)$ ,  $a_i$ , and  $v$  are distinct, and each set of the form  $D = \{s(i, j) : j \leq f(i)\}$  for some  $f : \omega \rightarrow \omega$  is a closed discrete subset of  $X$ . If the topology of  $X$  is determined by a countable family of compact subspaces  $\langle F_n : n \in \omega \rangle$  then the last requirement is equivalent to the condition that each set  $\{i : s(i, j) \in F_n \text{ for some } j \in \omega\}$ ,  $n \in \omega$  is finite. The point  $v$  will be called *the vertex* of  $s$ . It is well known (see [11]) that a free  $S_2$ -embedding exists for every countable  $k_\omega$  non discrete group and every countable sequential non Fréchet space.

The subspace  $S_2^- = \{x \in S_2 : |x| \in \{0, 2\}\}$  of  $S_2$  is a standard example of a non sequential subspace of a sequential space.

The space  $D_\omega$  (called  $\mathbb{L}$  in [6]) is defined as follows.  $D_\omega = \omega^2 \cup \{(\omega, \omega)\}$  where all the points in  $\omega^2$  are isolated and the base of open neighborhoods of  $(\omega, \omega)$  consists of  $(\omega \setminus n) \times \omega \cup \{(\omega, \omega)\}$ . The subspace  $(\omega + 1)^2 \setminus (\{\omega\} \times \omega \cup \omega \times \{\omega\})$  of the product of two convergent sequences is homeomorphic to  $D_\omega$ .

The diagonal  $\Delta = \{((m, n), \{m, m + n + 1\}) : m, n \in \omega\} \cup \{((\omega, \omega), \emptyset)\} \subseteq D_\omega \times S_2$  is a closed subset of the product homeomorphic to  $S_2^-$  (see [15], Remark 5.3). This example is an old result of van Douwen (see [6]).

## 4 $k_\omega$ groups

The following lemma is a group theoretic version of a result of E. van Douwen on the non productive nature of sequentiality mentioned at the end of Section 3. Just as the original result, its proof embeds a closed copy of  $S_2^-$  in the group.

**Lemma 3** (see [3], Lemma 4). *If  $G$  is a topological group that contains closed copies of both  $S(\omega)$  and  $D_\omega$  then  $G$  is not sequential.*

The result below is probably folklore. The version presented here does not require the knowledge of the topology on  $G$  in advance. The proof is given for the abelian case only, although the statement holds in a more general setting.

**Lemma 4.** *Let  $G$  be a group, and  $\langle F_n : n \in \omega \rangle$  be a cover of  $G$ . Suppose each  $F_n$  is given a compact Hausdorff topology such that for any  $i, j \in \omega$  the set  $F_i \cap F_j$  is closed in both  $F_i$  and  $F_j$  and the induced topologies are the same. Suppose further that the sums, inverses, and unions of any finite number of  $F_i$ 's are contained in some (possibly different)  $F_n$ 's and that the addition and algebraic inverse maps restricted to the corresponding (products of) compacts are continuous (for any large enough compact range). Then the topology  $\tau$  determined by  $\langle F_n : n \in \omega \rangle$  on  $G$  is a Hausdorff group topology.*

*Proof.* Since  $\tau$  is easily seen to be  $T_1$  it is enough to show that  $\tau$  is a group topology on  $G$ . Since  $\langle F_n : n \in \omega \rangle$  covers  $G$ ,  $\tau$  is invariant with respect to translations.

Let  $U$  be a subset of  $G$  such that  $0 \in U$  and  $U \cap F_n$  is open for every  $n \in \omega$ . Build, by induction on  $n \in \omega$ , a family  $V_n$ , such that:

- (1)  $V_n$  is an open subset of  $\cup_{i \leq n} F_i$  such that  $0 \in V_n \subseteq \overline{V_n} \subseteq U$ ;

$$(2) \overline{V_n} - \overline{V_n} \subseteq U$$

$$(3) \overline{V_m} \subseteq V_n \text{ for } m < n.$$

Assume  $F_0 = \{0\}$  and suppose  $V_i$  have been constructed for  $i < n$ . Since  $\overline{V_{n-1}}$  is a compact subset of  $\cup_{i \leq n} F_i$  and  $- : F_m^2 \rightarrow F_k$  is continuous (as well as each embedding  $F_i \subseteq F_m$ ), where  $\overline{F_m} \supseteq \cup_{i \leq n} F_i$ ,  $\overline{F_k} \supseteq \overline{F_m} - F_m$ , one can find an open  $V_n \subseteq \cup_{i \leq n} F_i$  such that  $\overline{V_{n-1}} \subseteq V_n$  and  $\overline{V_n} - \overline{V_n} \subseteq U$ . Now (1)–(3) are immediate so put  $V = \cup \{V_n : n \in \omega\}$ . Then (1) and (3) imply that  $V$  is open in  $\tau$  and (2) and (3) show that  $V - V \subseteq U$ .  $\square$

As a simple application of the lemma above observe that given a topological group  $G$  and a countable family  $\mathcal{K}$  of compact subspaces of  $G$  the finest group topology on  $G$  that induces the original topology on each  $K \in \mathcal{K}$  is  $k_\omega$ . For a proof simply consider the ‘algebraic closure’ of  $\mathcal{K}$ .

The next lemma is not stated in full generality (see Remark 1) as we are only interested in its applications in the more narrow setting below.

**Lemma 5.** *Let  $G$  be a boolean  $k_\omega$  group and  $D \subseteq G$  be an infinite closed discrete subset of  $G$ . Let  $a \in G$ . Then there exists an infinite subset  $C \subseteq D$  such that the finest group topology on  $G$  which is coarser than the original topology on  $G$  and such that  $C \rightarrow a$  is a  $k_\omega$  Hausdorff topology on  $G$ .*

*Proof.* Let  $\langle F_n : n \in \omega \rangle$  be a collection of compact subspaces of  $G$ , closed under finite unions, sums, and intersections that determines the topology of  $G$  (in particular, such a family is always a cover for  $G$ ). Since translations are homeomorphisms, assume  $a = 0$  by translating  $D$  if necessary.

Let  $D = \langle d_n : n \in \omega \rangle$ . Choose, by induction on  $i \in \omega$ , a sequence  $\langle n_i : i \in \omega \rangle$  such that for  $c_i = d_{n_i}$ ,  $C_i = \{c_j : j \leq i\} \cup \{0\}$ , and  $m > i$

$$(\cup_{j \leq i} F_j + \sum_{i=0}^i C_i) \cap (c_m + \cup_{j \leq i} F_j + \sum_{i=0}^i C_i) = \emptyset.$$

Since  $(\cup_{j \leq i} F_j + \sum_{i=0}^i C_i) - (\cup_{j \leq i} F_j + \sum_{i=0}^i C_i)$  is compact, and  $D$  is closed discrete, such a choice of  $n_i$  is possible. For convenience assume  $c_0 = 0 \in D$ .

Put  $C = \langle c_i : i \in \omega \rangle$ ,  $S_k = \sum_{i=0}^k C_i$ , and consider the natural addition map  $s_{k,n} : C^k \times F_n \rightarrow S_k + F_n$  defined as  $s_{k,n}(c^1, \dots, c^k, x) = c^1 + \dots + c^k + x$ . Introduce a natural topology on  $C$  such that  $C \rightarrow 0$  and view  $s$  as a quotient map onto its image. Suppose that with such a topology the image,  $S_k + F_n$ , is Hausdorff for all  $k < K$  and all  $n \in \omega$ . Choose an arbitrary  $n \in \omega$  and set  $i = \max\{K, n\}$ . To simplify the notation put  $s = s_{K,n}$ ,  $S = S_K$ . Note that the family  $(C_i^-)^K$ ,  $E_{j,l}$ ,  $j, l \leq i$  forms a clopen cover of  $C^K$  where  $C_i^- = (C \setminus C_i) \cup \{0\}$  and  $E_{j,l} = C \times \dots \times \{c_l\} \times \dots \times C$  where the  $j$ -th factor is equal to  $\{c_l\}$  while the remaining  $K - 1$  factors are  $C$ . Consider another addition map  $s^+ : (C_i^-)^K \times (F_n + \sum_{i=0}^K C_i) \rightarrow \sum_{i=0}^K C_i^- + F_n + \sum_{i=0}^K C_i$ . Note that  $s(C^K \times F_n) \subseteq s^+((C_i^-)^K \times (F_n + \sum_{i=0}^K C_i))$ . The map  $s^+$  can be further written as  $s^+(c^1, \dots, c^K, x) = s_2(s_1(c^1, \dots, c^K), x)$  where both  $s_1 : (C_i^-)^K \rightarrow \sum_{i=0}^K C_i^-$  and  $s_2 : (\sum_{i=0}^K C_i^-) \times (F_n + \sum_{i=0}^K C_i)$  are again additions.

Let  $c' + x' = c'' + x''$  where  $c', c'' \in \sum^K C_i^-$  and  $x', x'' \in F_n + \sum^K C_i$ . Choose (using the assumption that  $G$  is boolean) some representations  $c' = c'_1 + \dots + c'_p$  and  $c'' = c''_1 + \dots + c''_q$  where  $c'_j = c_{\nu(j)}$  and  $c''_j = c_{\mu(j)}$ , both  $\nu$  and  $\mu$  are strictly increasing and  $\nu(\cdot) > i, \mu(\cdot) > i$ . Put  $r = \max\{p, q\}$ . Show that  $c' = c''$  by an induction on  $r$ . If  $\nu(p) \neq \mu(q)$ , say  $\nu(p) > \mu(q)$ , then by the choice of  $c_{\nu(p)}$  the intersection  $(c_{\nu(p)} + F_n + \sum^K C_{\nu(p-1)}) \cap (F_n + \sum^K C_{\mu(q)})$  is empty but  $c' + x' \in c_{\nu(p)} + F_n + \sum^K C_{\nu(p-1)}$  and  $c'' + x'' \in F_n + \sum^K C_{\mu(q)}$ , contradicting  $c' + x' = c'' + x''$ . Thus  $\nu(p) = \mu(q)$ . If both  $p > 1$  and  $q > 1$ , replace  $c'$  with  $c' - c'_p$  and  $c''$  with  $c'' - c''_q$  and apply the inductive hypothesis. Note that  $p = 1$  if and only if  $q = 1$  by a similar argument implying  $c' = c''$ .

It follows that  $c' + x' = c'' + x''$  if and only if  $(c', x') = (c'', x'')$  for any  $(c', x'), (c'', x'') \in (\sum^K C_i^-) \times (F_n + \sum^K C_i)$ . Hence  $s_2$  is one-to-one and the (trivial) quotient topology it induces on its image is that of a (Cartesian) product of  $\sum^K C_i^-$  and  $F_n + \sum^K C_i$ . Note that  $F_n + \sum^K C_i$  is Hausdorff since  $F_n$  is a compact subset of the group and  $\sum^K C_i$  is finite. The Hausdorffness of  $\sum^K C_i^-$  (and the continuity of  $s_1$ ) can be established directly or by a shorter indirect argument given at the end of this proof. It remains to show that  $s$ , restricted to each  $(C_i^-)^K \times F_n$  and  $E_{j,l} \times F_n$  is continuous in the topology induced by  $s_2$ . The restriction of  $s$  to  $(C_i^-)^K \times F_n$  is the same as  $s^+$  restricted to the same set. Since  $s^+$  factors through  $s_1$  and  $s_2$ , both of which are continuous, the continuity of  $s$  on  $(C_i^-)^K \times F_n$  follows.

The continuity of  $s$  restricted to  $E_{j,l} \times F_n$  would follow from the continuity of the natural additive map  $s_3 : C^{K-1} \times (F_n + c_l) \rightarrow \sum^{K-1} C + F_n + c_l \subseteq \sum^K C_i^- + F_n + \sum^K C_i$ . To establish the continuity of  $s_3$ , observe, similar to an argument above, that  $s_3(C^{K-1} \times (F_n + c_l)) \subseteq s'_3((C_i^-)^{K-1} \times (F_n + \sum^K C_i))$  where  $s'_3$  is another addition. Just as above,  $s'_3$  can be written as  $s'_3(c^1, \dots, c^{K-1}, x) = s'_2(s'_1(c^1, \dots, c^{K-1}), x)$  where both  $s'_1 : (C_i^-)^{K-1} \rightarrow \sum^{K-1} C_i^-$  and  $s'_2 : (\sum^{K-1} C_i^-) \times (F_n + \sum^K C_i) \rightarrow \sum^{K-1} C_i^- + F_n + \sum^K C_i \subseteq \sum^K C_i^- + F_n + \sum^K C_i$  are additions. Identical to the proof for  $s_2$ , the map  $s'_2$  is one-to-one. Therefore, its continuity follows from the continuity of the embedding  $\sum^{K-1} C_i^- \subseteq \sum^K C_i^-$  where the topology of each sum is the quotient topology induced by the addition (see below for the proof). Thus  $s'_3$  is continuous and it remains to show that  $s_3$  is continuous when the topology of its range  $\sum^{K-1} C + F_n + c_l$  is inherited from the quotient topology on  $\sum^{K-1} C_i^- + F_n + \sum^K C_i$  induced by  $s'_3$ . This follows from the induction hypothesis applied to the addition  $s' : C^{K-1} \times F_o \rightarrow \sum^{K-1} C + F_o$  where  $o \in \omega$  is large enough so that  $F_n + \sum^K C_i \subseteq F_o$ . Indeed, both  $s_3$  and  $s'_3$  are restrictions of  $s'$ , the rest is the consequence of the uniqueness of a compact Hausdorff topology.

To show the Hausdorffness of the range of each  $g : C^K \rightarrow \sum^K C$ , observe, that (viewing  $G$  as a vector space over  $\mathbb{F}_2$ ) the set  $C \setminus \{0\}$  is linearly independent. It is also easy to show that if  $C'' \subseteq G'$  is an infinite convergent sequence in some boolean group  $G'$  then  $C' \subseteq C''$  for some infinite linearly independent set  $C'$ . Any one-to-one correspondence between  $C'$  and  $C \setminus \{0\}$  is now easily seen to induce the homeomorphism between  $\sum^K (C' \cup \{0\})$  as a subspace of  $G'$  and

$\sum^K C$  in the corresponding quotient topology.

Finally, the family  $\{s_{i,j}(C^i \times F_j) : i, j \in \omega\}$  satisfies all the properties of Lemma 4 and is easily seen to induce a topology on  $G$  that satisfies the conclusion of the lemma.  $\square$

**Remark 1.** The conditions on  $G$  and  $D$  in Lemma 5 can be weakened to requiring that  $D$  contain a  $T$ -sequence (see [12] for the definition and various properties and applications of  $T$ -sequences) and have the property that each  $mD$  is a closed discrete subset of  $G$  (in fact an even weaker condition would suffice).

Paper [2] mentions without proof that any non closed subgroup of a countable  $k_\omega$  group is not sequential. Below we present a proof of a generalization of this statement to uncountable groups.

**Lemma 6.** *Let  $G$  be a  $k_\omega$  group such that  $\psi(G) = \aleph_0$  and  $G$  is not Fréchet. Then  $G$  contains a closed copy of  $S(\omega)$ .*

*Proof.* Let  $\langle F_n : n \in \omega \rangle$  be a family of compact subspaces that determines the topology of  $G$ . Since  $\psi(G) = \aleph_0$  each  $F_n$  is first countable and  $G$  is sequential. Since  $G$  is not Fréchet, there exists an injective map  $s : \omega^2 \rightarrow G$  such that  $s(i, j) \rightarrow a_i$  where  $a_i \rightarrow v$  and for any  $f : \omega \rightarrow \omega$  the sequence  $s(i, f(i)) \not\rightarrow v$ . Since each  $F_n$  is first countable, the set  $\{i : s(i, j) \in F_n \text{ for infinitely many } j \in \omega\}$  is finite for every  $n \in \omega$ . By thinning out  $s$  if necessary, one can assume that each set of the form  $\{i : s(i, j) \in F_n \text{ for some } j \in \omega\}$  is finite. If  $a_i = v$  for infinitely many  $i \in \omega$ , it is easy to see that there is a closed copy of  $S(\omega)$  in  $\overline{s(\omega^2)} = s(\omega^2) \cup \langle a_i : i \in \omega \rangle \cup \{v\}$ . Otherwise the same set contains a closed copy of  $S_2$  so (see [11])  $G$  contains a closed copy of  $S(\omega)$ .  $\square$

Let us mention without proof a result about the sequential order of  $k_\omega$ -groups of countable pseudcharacter. The proof is an extension of the techniques presented here (see also Lemma 1 below).

**Lemma 7.** *Let  $G$  be a  $k_\omega$  group such that  $\psi(G) = \aleph_0$  and  $G$  is not Fréchet. Then  $\text{so}(G) = \omega_1$ .*

It turns out that a  $k_\omega$  group can be ‘reduced’ by taking an appropriate quotient.

**Lemma 8.** *Let  $G$  be a separable sequential non Fréchet  $k_\omega$  group. Then there exists a closed normal subgroup  $H \subseteq G$  of  $G$  such that  $G/H$  is not Fréchet and  $\psi(G/H) = \aleph_0$ .*

*Proof.* Suppose  $G$  is not Fréchet. Then there exists a countable subgroup  $G' \subseteq G$  that is not Fréchet. Pick a countable stable family  $\mathcal{U}$  of open subsets of  $G$  such that  $p$  is one-to-one on  $G'$  and  $p(G' \cap F_n) = p(G') \cap p(F_n)$  for every  $n \in \omega$  where  $p : G \rightarrow G/H$  is a natural quotient map and  $H = \bigcap \mathcal{U}$ . Note that  $\psi(G/H) = \aleph_0$ . If  $G/H$  is Fréchet, as a sequential  $k_\omega$  group it is locally compact so let  $1 \in U \subseteq G/H$  be an open subset of  $G/H$  such that  $\overline{U}$  is compact. Now



$V = p^{-1}(U) \cap \overline{G'}$  is an open neighborhood of 1 in  $\overline{G'}$ . If  $\overline{V}$  is not (countably) compact there exists a closed infinite discrete subset  $D \subseteq \overline{V}$ . Note that for every  $n \in \omega$  the intersection  $F_n \cap D$  is finite and  $V \cap G'$  is dense in  $\overline{V}$  so one can pick an infinite subset  $D' \subseteq V \cap G'$  such that every intersection  $F_n \cap D'$  is finite. Hence every  $p(F_n) \cap p(D') = p(F_n \cap D')$  is finite and  $p(D')$  is infinite. Therefore  $p(D')$  is an infinite closed discrete subset of  $\overline{U}$ , a contradiction. Thus  $\overline{G'}$  is locally compact and thus Fréchet, contradicting the choice of  $G'$ .  $\square$

As a corollary of the lemma above and Lemma 7 one shows

**Proposition 1.** *Let  $G$  be a sequential  $k_\omega$  group. Then either  $G$  is locally compact (and therefore metrizable) or  $\text{so}(G) = \omega_1$ .*

*Proof.* Note that an open map cannot raise the sequential order by Lemma 2 then apply Lemmas 7 and 8.  $\square$

It is possible, in fact, to obtain a closed embedding  $S_\omega \subseteq G'$  for any dense subgroup  $G'$  in this case. Lemma 7 and Proposition 1 will not be used in this paper.

**Proposition 2.** *Let  $G$  be a topological group and  $G' \subseteq G$  be such that  $\overline{G'}$  is a sequential non Fréchet  $k_\omega$  group. Then  $G'$  contains a copy of  $S(\omega)$  closed in  $G$ .*

*Proof.* Since  $G'' = \overline{G'}$  is not Fréchet there exists a countable  $G''' \subseteq G'$  such that  $\overline{G'''} is not Fréchet so we may assume that  $G'$  is countable and  $G = \overline{G'}$ . Using Lemma 8 and Lemma 6, find a closed subgroup  $H \subseteq G$  such that  $\psi(G/H) = \aleph_0$ ,  $p(G' \cap F_n) = p(G') \cap p(F_n)$  for every  $n \in \omega$ , and  $G/H$  is not Fréchet.$

Suppose first there exists an injective map  $s : \omega^2 \rightarrow p(G')$  such that  $s(i, j) \rightarrow a_i$  for some  $a_i \rightarrow v$  and  $s(i, f(i)) \not\rightarrow v$  for any  $f : \omega \rightarrow \omega$ . Just as in Lemma 6 one can thin out  $s$  if necessary to assume that  $s$  is a free  $S_2$ -embedding into  $G/H$ . Let  $s'(i, j)$  be the only point in  $p^{-1}(s(i, j)) \cap G'$ . The set  $\{s'(i, j) : j \in \omega\}$  cannot be closed discrete in  $G$ . Indeed, for some  $n \in \omega$  the set  $p(F_n) \cap \{s(i, j) : j \in \omega\} = p(F_n \cap \{s'(i, j) : j \in \omega\})$  is infinite. Thinning  $s$  again, if necessary one may assume that  $s'(i, j) \rightarrow a'_i \in p^{-1}(a_i)$ .

Given  $n \in \omega$  find an  $i_n \in \omega$  such that  $p(\cup_{i \leq n} F_i) \cap \{s(i_n, j) : j \in \omega\} = \emptyset$ . Note that  $G' \ni s'(i_n, j) \cdot (s'(i_n, k))^{-1} \rightarrow s'(i_n, j) \cdot (a'_{i_n})^{-1}$  and  $s'(i_n, j) \cdot (a'_{i_n})^{-1} \rightarrow 0$  and choose strictly increasing sequences  $k_i, j_i \in \omega$  such that  $s''(n, i) = s'(i_n, j_i) \cdot (s'(i_n, k_i))^{-1} \notin \cup_{l \leq n} F_l$  and  $s'' : \omega^2 \rightarrow G'$  is injective. Observe that  $s''(i, j) \rightarrow 0$  as  $j \rightarrow \infty$  and each set  $\{i : s''(i, j) \in F_n \text{ for some } j \in \omega\}$  is finite. Thus  $s''(\omega^2) \cup \{0\}$  is a closed copy of  $S(\omega)$  in  $G'$ .

If such  $s$  does not exist, it follows that for every  $a \in \overline{p(G')}$  there exists a sequence  $\langle c_n : n \in \omega \rangle \subseteq p(G')$  such that  $c_n \rightarrow a$ . Using Lemma 6 find a closed copy of  $S(\omega)$  in  $G/H$ . We can assume that the map  $s : \omega^2 \rightarrow G/H$  that witnesses this embedding is such that  $s(i, j) \rightarrow 0$  as  $j \rightarrow \infty$  and each  $\{i : s(i, j) \in p(F_n) \text{ for some } j \in \omega\}$  is finite. As before find an  $i_n \in \omega$  such that  $p(F_n) \cap \{s(i_n, j) : j \in \omega\} = \emptyset$ . Pick a point  $c_i(j) \in p(G') \setminus p(\cup_{l \leq n} F_l)$  for each  $i, j \in \omega$  so that  $c_i(j) \rightarrow s(i_n, j)$ . By our assumption there exist strictly increasing sequences  $k_i, j_i \in \omega$  such that  $s'(n, i) = c_{k_i}(j_i) \rightarrow 0$ . By the choice of

$i_n$  we may assume that  $s'(\omega^2) \cup \{0\}$  is a closed copy of  $S(\omega)$  in  $p(G')$ . Repeating the argument of the previous paragraph verbatim, one can find a closed copy of  $S(\omega)$  in  $G'$ .  $\square$

**Lemma 9.** *Let  $G' \subseteq G$  be a subgroup of a sequential group  $G$  which is not closed in  $G$ . Then  $G'$  contains a closed copy of  $D_\omega$ .*

*Proof.* There exists a point  $a \in G \setminus G'$  and a sequence  $\langle c_i : i \in \omega \rangle \subseteq G'$  such that  $c_i \rightarrow a$ . Note that  $c_i \cdot c_j^{-1} \rightarrow c_i \cdot a^{-1} \notin G'$  and  $c_i \cdot a^{-1} \rightarrow 1$ . Therefore  $G' \cap (\langle c_i : i \in \omega \rangle \cup \{a\}) \cdot (\langle c_i : i \in \omega \rangle \cup \{a\})^{-1}$  is a first countable closed subspace of  $G'$  that is not locally compact at 1. A standard argument produces a closed copy of  $D_\omega$  in  $G'$ .  $\square$

Finally, the generalization promised before Lemma 6 is a direct corollary of Lemma 3, Proposition 2, and Lemma 9.

**Proposition 3.** *Let  $G$  be a sequential  $k_\omega$  group and  $G' \subseteq G$  be a subgroup of  $G$  such that  $\overline{G'}$  is not Fréchet and  $\overline{G'} \neq G'$ . Then  $G'$  is not sequential.*

Note that the condition that  $\overline{G'}$  is not Fréchet cannot be dropped above as the standard embeddings  $\mathbb{Q} \subseteq \overline{\mathbb{Q}} = \mathbb{R} \subseteq \mathbb{R}^\infty$  show. Here  $\mathbb{R}^\infty$  is the direct limit of  $\mathbb{R}^n$ 's.

As a simple application of Proposition 3, consider the following example that indicates that sequentiality may not be readily inherited by countable subgroups (unlike Fréchetness) even in the  $k_\omega$  case. Note that similar topologies have been considered before (see, e.g. [10]), although in a different context.

**Example 1.** *A  $k_\omega$  sequential topology on  $\mathbb{R}$  such that the only proper sequential subgroups of  $\mathbb{R}$  are closed cyclic.*

Applying the stronger version of Lemma 5 mentioned in Remark 1, one can construct real numbers  $\langle r_n : n \in \omega \rangle$  such that  $r_n \rightarrow \infty$  and a  $k_\omega$  group topology on  $\mathbb{R}$  coarser than the original topology and such that it is the finest group topology in which  $r_n \rightarrow 0$ . There is some freedom in choosing the algebraic properties of  $r_n$ 's as well. Thus one can assume that all  $r_n$ 's are integers, or linearly independent over  $\mathbb{Q}$ . It is easy to see that in such a topology  $\mathbb{R}$  becomes a sequential non Fréchet group (indeed, its sequential order is  $\omega_1$ , see [14]). It is also an easy observation that any countable closed subgroup of  $\mathbb{R}$  must be cyclic and that every cyclic subgroup of  $\mathbb{R}$  is dense in itself in the new topology. Thus, by Proposition 3, the only possible sequential subgroups of  $\mathbb{R}$  in this topology are cyclic and  $\mathbb{R}$  itself.

Given any countably many cyclic subgroups of  $\mathbb{R}$ , it is not difficult to pick a sequence as above that would make each one of the subgroups not closed (and thus not sequential by Proposition 3). The question whether such a sequence can make *all* nontrivial subgroups of  $\mathbb{R}$  cease to be sequential seems to require some number theoretic tools (such as deeper understanding of Kronecker sequences  $n\alpha \bmod 1$  for an irrational  $\alpha$ ) the author currently lacks. Note that Theorem 3.1 of [5] implies that the sequence as above, consisting of *integers*

must have  $\limsup r_{n+1}r_n^{-1} < \infty$  for the set of  $\alpha \in \mathbb{R}$  such that  $r_n\alpha \rightarrow 0$  (topologically torsion elements) to be countable. Since this property is not satisfied by the construction in Lemma 5, a single integer sequence  $\langle r_n : n \in \omega \rangle$  is not enough to ensure that every cyclic subgroup is not closed in this topology. It is unclear whether non integer sequences would have this property, or if a countable number of sequences can provide the required properties (i.e. make every cyclic subgroup dense in  $\mathbb{R}$ ).

The next lemma shows that one can add a free  $S_2$ -embedding that witnesses the nonsequentiality of a given subgroup without disturbing countably many other such embeddings. The requirement that  $G$  is boolean and co-countable can be weakened significantly at the expense of a longer proof.

**Lemma 10.** *Let  $G$  be a boolean co-countable sequential  $k_\omega$  group,  $G'$  be a countable nondiscrete subgroup of  $G$ . Let  $\mathcal{S}$  be a countable family of free  $S_2$ -embeddings into  $G$ . Let  $\mathcal{U}$  be a countable stable family of open subsets of  $G$ . Then there exists a  $k_\omega$  group topology  $\tau^\infty$  on  $G$ , coarser than the original topology, and a free  $S_2$ -embedding  $s' : \omega^2 \rightarrow G'$  with respect to  $\tau^\infty$  such that (1) each  $s \in \mathcal{S}$  is a free  $S_2$ -embedding into  $G$  with respect to  $\tau^\infty$ ; (2) each  $U \in \mathcal{U}$  is open in  $\tau^\infty$ ; and (3)  $s'(i, j) \rightarrow a_i \notin G'$  for some  $a_i \rightarrow 0$  in  $\tau^\infty$ . Moreover  $\tau^\infty$  is the finest group topology on  $G$  coarser than its original topology and such that  $S_i \rightarrow a_i$  for some  $\langle S_i : i \in \omega \rangle$ .*

*Proof.* Suppose the topology of  $G$  is determined by a countable family  $\langle F_n : n \in \omega \rangle$  of compact subsets closed with respect to finite intersections, finite unions, and natural algebraic operations. For the sake of simplicity, assume that  $\mathcal{S} = \{s\}$  where  $s : \omega^2 \rightarrow G$  is a free  $S_2$ -embedding. The case of a countable  $\mathcal{S}$  is similar. Let  $G'$  be an arbitrary countable subgroup. Use Lemma 1 to extend  $\mathcal{U}$  to a countable family  $\mathcal{U}'$  of open subsets of  $G$  such that  $\mathcal{U}'$  is stable,  $H = \bigcap \mathcal{U}'$  is a closed subgroup of  $G$ ,  $p$  is one-to-one on  $G' \cup s(\omega^2)$  and for any  $n \in \omega$  the set  $\{i \in \omega : p \circ s(i, j) \in p(F_n) \text{ for some } j \in \omega\}$  is finite. Now the family  $\langle p(F_n) : n \in \omega \rangle$  of compact subsets determines the topology of the countable group  $G/H$ , so  $p \circ s$  is a free  $S_2$ -embedding into  $G/H$ , and  $U = p^{-1}(p(U))$  for each  $U \in \mathcal{U}'$ . We may assume, by extending  $\mathcal{U}'$  if necessary, that  $\{p(U) : U \in \mathcal{U}'\}$  forms a basis of open neighborhoods for some first-countable group topology  $\tau_\omega$  coarser than the topology  $\tau$  induced by  $p$ .

Find a function  $f : \omega \rightarrow \omega$  such that  $K = \overline{p \circ s(\{(m, n) : n \geq f(n)\})}$  is compact in  $\tau_\omega$ . Since the set  $D = \{p \circ s(m, n) : n < f(m)\}$  is closed and discrete in  $\tau$  we can extend  $\tau_\omega$  to a first countable topology  $\tau'_\omega$  coarser than  $\tau$  so that  $D$  is closed and discrete in  $\tau'_\omega$ .

If  $G'$  is not discrete, its image  $p(G')$  is dense in itself in  $\tau$ ,  $\tau_\omega$ , and  $\tau'_\omega$ . Therefore, one can find a countable family  $\langle V_n : n \in \omega \rangle$  of (relatively) open in  $\tau'_\omega$  subsets of  $p(G')$  such that  $\langle V_n : n \in \omega \rangle \rightarrow 0$  in  $\tau'_\omega$ . Note that if  $C = \langle c_n : n \in \omega \rangle \cup \{a\}$  is chosen so that  $a \in p^{-1}(0)$  and  $c_n \in p^{-1}(V_n)$  then for any  $n, k \in \omega$  the intersection  $(F_n + \sum^k C) \cap D'$  is finite, where  $D' = \{s(m, n) : n < f(m)\}$ . Indeed  $p(C)$  is a compact subset of  $p(G')$  in  $\tau'_\omega$ ,  $p$  is one-to-one on  $D'$ , and  $p(D') = D$  is a closed discrete subset of  $G/H$  in  $\tau'_\omega$ .

Use induction on  $i \in \omega$  to find  $c'_i \in V_i$  such that for any  $m > i$

$$(c'_m + \cup_{j \leq i} p(F_j) + \sum_{i=0}^i C_i) \cap (K \cup (\cup_{j \leq i} p(F_j))) = \emptyset$$

where  $C_i = \{c'_j : j \leq i\} \cup \{0\}$ . Since each  $V_n$  is dense in itself (in both  $\tau'_\omega$  and  $\tau_\omega$ ), while  $K, p(F_j)$ 's and their sums are compact in  $\tau_\omega$  and thus scattered, such a choice of  $c_i$  is possible.

Now inductively assume that for any  $n \in \omega$  and any  $m < M$  the set

$$\{j : p \circ s(j, l) \in p(F_n) + \sum_{i=0}^m C_i \text{ for some } l \geq f(j)\}$$

is finite. Pick  $i = \max\{M, n\}$ . Note that the set  $p(F_n) + \sum_{i=0}^M C_i$  is a union of  $p(F_n) + \sum_{i=0}^M C_i^-$  and finitely many sets of the form  $p(F_n) + \sum_{i=0}^{M-1} C_i + c'_l$  for some  $l \leq i$ , where  $C_i^- = \{c'_j : j > i\} \cup \{0\}$ . By the induction hypothesis it is enough to show that the set

$$\{j : p \circ s(j, l) \in p(F_n) + \sum_{i=0}^M C_i^- \text{ for some } l \geq f(j)\}$$

is finite. To see this, observe that for  $c \in \sum_{i=0}^M C_i^-$  either  $c = 0$  or one can write  $c = c^1 + \dots + c^p$  where  $c^j = c'_{\nu(j)}$  such that  $\nu$  is strictly increasing and  $\nu(j) > i$ . In the latter case, by the choice of  $c'_i$  the intersection  $(c'_{\nu(p)} + p(F_n) + \sum_{i=0}^{\nu(p)-1} C_{i(p)-1}^-) \cap K = \emptyset$  so

$$\begin{aligned} & \{j : p \circ s(j, l) \in p(F_n) + \sum_{i=0}^M C_i^- \text{ for some } l \geq f(j)\} \\ &= \{j : p \circ s(j, l) \in p(F_n) \text{ for some } l \geq f(j)\}. \end{aligned}$$

The last set is finite since  $p(F_n)$  is compact and  $p \circ s$  is a free  $S_2$ -embedding in  $\tau$ .

Let  $c_i \in G' \cap p^{-1}(c'_i)$ ,  $S = \langle c_i : i \in \omega \rangle$ . It is immediate that  $S$  is a closed discrete subset of  $G$  such that for every  $n, k \in \omega$  the set

$$\{j : s(j, l) \in F_n + \sum_{i=0}^k S \text{ for some } l \in \omega\}$$

is finite. Moreover, since  $p(S) \rightarrow 0$  in  $\tau_\omega$ ,  $S \subseteq^* U$  for every  $U \in \mathcal{U}'$ .

Choose an infinite sequence  $\langle a_i : i \in \omega \rangle \subseteq H$  such that  $a_i \rightarrow 0$ . Use Lemma 5 to find an  $S_0 \subseteq S$  such that  $S \rightarrow a_0$  in a  $k_\omega$ -topology  $\tau'$  on  $G$  coarser than the original topology and such that  $\tau'$  is the finest group topology on  $G$  with these properties. Note that with respect to  $\tau'$  both  $\mathcal{U}$  and  $\mathcal{U}'$  remain stable countable families of open subsets of  $G$ . Thus the argument above can be repeated using  $\tau'$  instead of the original topology on  $G$  to obtain  $S_1 \rightarrow a_1$  and  $\tau''$ . Eventually one can construct  $k_\omega$ -topologies  $\tau^{(i)}$  and sequences  $S_i \rightarrow a_i$  in  $\tau^{(i)}$  such that (i)  $\mathcal{U}$  is a countable stable family of subsets of  $G$  open in  $\tau^\infty = \cap_i \tau^{(i)}$ ; (ii)  $\tau^{(i+1)}$

is the finest group topology on  $G$  coarser than  $\tau^{(i)}$  in which  $S_i \rightarrow a_i$ ; (iii) each  $\tau^{(i)}$  is  $k_\omega$ ; (iv)  $S_i$  is a closed discrete subset of  $G$  in  $\tau^{(j)}$  for all  $i \geq j$ ; (v)  $s$  is a free  $S_2$ -embedding in  $\tau^\infty$ .

Let now  $\langle K_i : i \in \omega \rangle$  be a family of compact subsets of  $G$  in  $\tau^\infty$  that determines  $\tau^\infty$ . Let  $S_i = \langle c_j^i : j \in \omega \rangle$  and put  $s'(j, l) = c_l^j$ . We can assume that  $s'$  is one-to-one and (using (iv) above) that each  $K_n$  intersects at most finitely many  $S_i$ 's. Thus  $s'$  is a free  $S_2$ -embedding into  $G$  in  $\tau^\infty$ .  $\square$

**Remark 2.** As an alternative to building  $S_i$ 's at the end of the proof above one could show that after adding  $S_0$  the closure of  $G'$  in  $G$  will not be Fréchet and then use the proof of Proposition 3 that produces a closed copy of  $S_2^-$  in  $G'$ .

The next lemma is used to make the quotients of  $G$  Fréchet.

**Lemma 11.** *Let  $G$  be a boolean co-countable  $k_\omega$  group and let  $\mathcal{U}$  be a countable stable family of open subsets of  $G$ . Let  $H = \cap \mathcal{U}$  be a closed subgroup of  $G$  such that  $H \not\subseteq F + E$  for any compact  $F \subseteq G$  and any countable subset  $E \subseteq G$ . Let  $\sigma : \omega^2 \rightarrow G/H$  be a free  $S_2$ -embedding in the topology induced by the natural quotient map  $p$  and let  $\mathcal{S}$  be a countable family of free  $S_2$ -embeddings into  $G$ . Then there exists a countable subset  $S \subseteq G$  such that each  $U \in \mathcal{U}$  is open in the topology  $\tau$  which is the finest topology coarser than the original topology of  $G$  and such that  $S \rightarrow v$  for some  $v \in G$ , each element of  $\mathcal{S}$  is a free  $S_2$ -embedding in  $\tau$  and  $\sigma$  is no longer a free  $S_2$ -embedding.*

*Proof.* To simplify notation, assume  $\mathcal{S} = \{s\}$  for some free  $S_2$ -embedding  $s : \omega^2 \rightarrow G$ . Let  $\langle F_n : n \in \omega \rangle$  be a countable family of compact subsets of  $G$  that determines the topology of  $G$ . Since  $\sigma$  is a free  $S_2$ -embedding into  $G/H$ , one can find a function  $f : \omega \rightarrow \omega$  such that the sequence  $\langle \sigma(n, f(n)) : n \in \omega \rangle$  converges to the vertex of  $\sigma$  in the first-countable group topology  $\tau_\omega$  generated by the base of open neighborhoods  $\{p(U) : U \in \mathcal{U}\}$ . By induction on  $i \in \omega$  find a sequence  $c_i \in p^{-1}(\sigma(i, f(i)))$  such that for any  $n \in \omega$ ,  $m > i$

$$(c_m + \cup_{j \leq i} F_j + \sum_{i=0}^m C_i) \cap (K \cup (\cup_{j \leq i} F_j)) = \emptyset$$

where  $C_i = \{c_j : j \leq i\}$ . Since  $K$  is countable, the condition imposed on the kernel of  $p$  makes such selection possible.

Just as in Lemma 10 one can show that  $S = \langle c_n : n \in \omega \rangle$  has the property that for any  $k, n \in \omega$  the set  $\{j : s(j, l) \in \sum^k S + F_n\}$  is finite and  $S \subseteq^* U$  for each  $U \in \mathcal{U}$ . It is also immediate that  $S$  is a closed discrete subset of  $G$ . Apply Lemma 5 to find a subset  $S' \subseteq S$  such that  $S' \rightarrow v \in p^{-1}(v_\sigma)$  where  $v_\sigma$  is the vertex of  $\sigma$ .  $\square$

## 5 Example

This is the main example in the paper.

**Example 2** ( $\diamond$ ). *A sequential group  $G$  such that every countable sequential subgroup of  $G$  is discrete and every quotient of  $G$  is either Fréchet or has an uncountable pseudocharacter.*

Let  $A \subseteq 2^{\omega_1}$  be a subspace homeomorphic to the one point compactification of a discrete space of size  $\omega_1$ . We can assume, by thinning out and translating  $A$  if necessary, that  $0$  is the only non isolated point of  $A$  and that all the isolated points are linearly independent over  $\mathbb{F}_2$ . Let  $G$  be the subgroup of  $2^{\omega_1}$  generated by  $A$ .

**Claim 1.** Let  $n \in \omega$ ,  $E \subseteq G$  be a countable subset of  $G$ . Let  $H$  be a subgroup of  $G$  generated by an uncountable subset  $B$  of  $A$ . Then  $\sum^n A + E$  does not cover  $H$ . Indeed, let  $E' \subset A$  be a countable set whose span contains  $E$ , and let  $a^1, \dots, a^{n+1} \in B$  be points of  $B$  that are not in the span of  $E'$ . Then  $a^1 + \dots + a^{n+1} \notin \sum^n A + E$ .

Finally, let  $\tau_0$  be the finest group topology on  $G$  that induces the original topology on  $A$ . By an observation after Lemma 4  $G$  is a co-countable  $k_\omega$ -group.

Given countable subsets  $S_i \subseteq G$ ,  $0 \leq i \leq k$ , and  $n \in \omega$ , consider the natural additive map  $a_{\{S_i\},k,n} : \prod_{i=0}^k (S_i \cup \{0\}) \times \prod^n A \rightarrow (\sum_{i=0}^k (S_i \cup \{0\})) + \sum^n A$ , viewed as a quotient map where  $S_i \cup \{0\}$  is given the unique topology such that  $S_i \rightarrow 0$ . Suppose  $a_{\{S_i\},k,n}$  induces a (compact) Hausdorff topology on its image. Then the image has weight  $\omega_1$  so let  $\{U_{\{S_i\},k,n}^\alpha : \alpha \in \omega_1\}$  be a base of open neighborhoods for the image. Now use CH to find  $B : \omega_1 \rightarrow 2^G$  that lists every  $U_{\{S_i\},k,n}^\alpha$  with the properties above. Fix a  $\diamond$ -sequence  $\{A_\alpha : \alpha \in \omega_1\}$ . Let  $\{G_\alpha : \alpha \in \omega_1\}$ ,  $\{\sigma_\alpha : \alpha \in \omega_1\}$  list every countable subgroup of  $G$  and every one-to-one map  $\sigma : \omega^2 \rightarrow G$  respectively, unboundedly many times.

Construct, by induction on  $\alpha \in \omega_1$ , decreasing  $k_\omega$  group topologies  $\tau_\alpha$  on  $G$ , one-to-one maps  $s_\alpha : \omega^2 \rightarrow G_\alpha$ , increasing countable families  $\mathcal{S}_\alpha$  of countable subsets of  $G$ , and increasing countable families  $\mathcal{U}_\alpha$  of open in  $\tau_\alpha$  subsets of  $G$  such that

- (a) each  $\tau_\alpha$  is the finest group topology on  $G$  that induces the original topology on  $A$  such that  $S \rightarrow 0$  for each  $S \in \mathcal{S}_\alpha$ .
- (b) if  $G_\alpha$  is not closed discrete in  $\tau_\alpha$  then  $s_\alpha$  is a free  $S_2$ -embedding into  $G$  with respect to  $\tau_{\alpha+1}$  such that  $s_\alpha(i, j) \rightarrow a_i \notin G_\alpha$  where  $a_i \rightarrow 0$ .
- (c) each  $s_\beta$  is a free  $S_2$ -embedding in  $\tau_\alpha$  for  $\beta < \alpha$ .
- (d) each  $\mathcal{U}_\alpha$  is stable.
- (e) if  $V = \cup\{B(\beta) : \beta \in A_\alpha\}$  is such that  $0 \in \text{Int}(V)$  in  $\tau_\alpha$  then there is  $U \in \mathcal{U}_\alpha$  such that  $0 \in U \subseteq \text{Int}(V)$ .
- (f)  $p_\alpha \circ \sigma_\alpha$  is not a free  $S_2$ -embedding in  $G/H_\alpha$  with respect to the topology induced by  $p_\alpha : G \rightarrow G/H_\alpha$  where  $H_\alpha = \cap \mathcal{U}_\alpha$  and the topology on  $G$  is  $\tau_\alpha$ .

Suppose  $\tau_\beta$ ,  $s_\beta$ ,  $\mathcal{U}_\beta$ , and  $\mathcal{S}_\beta$  satisfying (a)–(f) have been built for  $\beta < \alpha$ . Put  $\tau'_\alpha = \cap_{\beta < \alpha} \tau_\beta$ ,  $\mathcal{U}'_\alpha = \cup_{\beta < \alpha} \mathcal{U}_\beta$ , and  $\mathcal{S}'_\alpha = \cup_{\beta < \alpha} \mathcal{S}_\beta$ . Observe that each  $U \in \mathcal{U}'_\alpha$  is open in  $\tau'_\alpha$  and that  $\tau'_\alpha$  is the finest topology such that it induces the original topology on  $A$  and  $S \rightarrow 0$  for each  $S \in \mathcal{S}'_\alpha$ . If  $V = \cup\{B(\beta) : \beta \in A_\alpha\}$  has a nonempty interior in  $\tau'_\alpha$  and  $0 \in \text{Int}(V)$  extend  $\mathcal{U}'_\alpha$  to a countable family  $\mathcal{U}_\alpha$  of open subsets of  $G$  in  $\tau'_\alpha$  such that  $\mathcal{U}_\alpha$  is stable and  $0 \in U \subseteq \text{Int}(V)$  for some  $U \in \mathcal{U}_\alpha$ . If  $G_\alpha$  is not a closed discrete subgroup of  $G$  in  $\tau'_\alpha$ , use Lemma 10 to find a countable family  $\mathcal{S}$  of countable subsets of  $G$  and a one-to-one map  $s_\alpha : \omega^2 \rightarrow G_\alpha$  such that  $\{s_\beta : \beta \leq \alpha\}$  are free  $S_2$ -embeddings with respect to  $\tau''_\alpha$ , and each  $U \in \mathcal{U}_\alpha$  is open in  $\tau''_\alpha$ , where  $\tau''_\alpha$  is the finest group topology on  $G$  such that  $S \rightarrow 0$  for each  $S \in \mathcal{S}$ , coarser than  $\tau'_\alpha$ . Otherwise put  $s_\alpha = s_\beta$  for some  $\beta < \alpha$ ,  $\mathcal{S} = \emptyset$ , and  $\tau''_\alpha = \tau'_\alpha$ .

Now if  $p \circ \sigma_\alpha$  is a free  $S_2$ -embedding into  $G/H$  where  $p : G \rightarrow G/H$  is the natural quotient map in  $\tau''_\alpha$ , and  $H = \cap \mathcal{U}_\alpha$ , use Lemma 11 to find a convergent sequence  $S \subseteq G$  in a  $k_\omega$ -topology  $\tau_\alpha$  coarser than  $\tau''_\alpha$  and such that each  $U \in \mathcal{U}_\alpha$  is open in  $\tau_\alpha$ , and  $p \circ \sigma_\alpha$  is not a free  $S_2$ -embedding in  $\tau_\alpha/H$ .

That the conditions of Lemma 11 are satisfied follows from Claim 1 above and an easy observation that each compact subset of  $G$  in any  $\tau_\alpha$  is included in a union of countably many translations of a single sum of  $n$  copies of  $A$ , as follows from (a).

If  $p \circ \sigma_\alpha$  is not a free  $S_2$ -embedding take  $S$  to be an arbitrary convergent sequence in  $\tau''_\alpha$ , put  $\tau_\alpha = \tau''_\alpha$ .

Put  $\mathcal{S}_\alpha = \mathcal{S}'_\alpha \cup \mathcal{S} \cup \{S\}$ . (a)–(f) follow.

Let  $\tau = \cap_{\alpha < \omega_1} \tau_\alpha$ . It is immediate that  $\tau$  is a sequential topology on  $G$  invariant with respect to translations. Properties (a)–(f) imply that each  $s_\alpha$  is a free  $S_2$ -embedding with respect to  $\tau$ . Since each  $G_\alpha$  is listed unboundedly many times, if  $G_\alpha$  is not closed discrete in  $\tau$  it is not closed discrete in some  $\tau_\beta$  such that  $G_\alpha = G_\beta$ . Then (b) ensures that the topology inherited by  $G_\alpha$  from  $\tau$  is not sequential.

To show that  $\tau$  is a group topology put  $\mathcal{U} = \cup_\alpha \mathcal{U}_\alpha$ . Since each  $U \in \mathcal{U}$  is open in  $\tau$  ( $\{\mathcal{U}_\alpha\}$  is increasing and each  $U \in \mathcal{U}_\alpha$  is open in  $\tau_\alpha$ ) and  $\mathcal{U}$  is stable it is enough to show that  $\mathcal{U}_\alpha$  is a basis of open neighborhoods of  $0 \in G$  in  $\tau$ .

Let  $F \subseteq G$  be a subset of  $G$  closed in  $\tau$  such that  $0 \notin F$ . Let  $\alpha < \omega_1$  and  $O \subseteq \omega_1$  be the set of all  $\beta \in \omega_1$  such that  $\overline{B(\beta)}$  is an open subset of some compact  $K \subseteq G$  in  $\tau_{t(\beta)}$  for  $t(\beta) > \beta$  and  $\overline{B(\beta)} \cap F = \emptyset$  where the closure is taken in  $\tau_{t(\beta)}$ . Put  $O_\alpha = O \cap \alpha$  and call  $\alpha$  *F-saturated in  $\tau$*  if the following conditions are met:

- (1) if  $\beta_1, \dots, \beta_n \in O_\alpha$  and  $K \subseteq G$  is compact (in  $\tau_\alpha$ ) then there exist  $\beta^1, \dots, \beta^k \in O_\alpha$  such that each  $\overline{B(\beta^i)} \cap K$  is relatively open in  $K$ , and  $\cup_{i \leq n} \overline{B(\beta_i)} \subseteq \cup_{j \leq k} \overline{B(\beta^j)} \subseteq G \setminus F$ .
- (2) for each  $\beta \in O_\alpha$   $t(\beta) < \alpha$  and there exists  $\beta \in O_\alpha$  such that  $0 \in B(\beta)$ .

Note that an inductive construction similar to that of Lemma 4 shows that  $0 \in \text{Int}_{\tau_\alpha}(\cup_{\beta \in O_\alpha} B(\beta)) \subseteq G \setminus F$  and a standard argument shows that the ordinals  $F$ -saturated in  $\tau$  form a club. Therefore there exists an  $\alpha \in \omega_1$  such that

$A_\alpha = O_\alpha$  and  $0 \in \text{Int}(A_\alpha) \subseteq F$  in  $\tau_\alpha$  so by (e) above there is a  $U \in \mathcal{U}_\alpha \subseteq \mathcal{U}$  such that  $0 \in U \subseteq G \setminus F$ .

Suppose  $H \subseteq G$  is a closed (in  $\tau$ ) subgroup of  $F$  such that  $G/H$  is not Fréchet and  $\psi(G/H) = \omega$ . Since  $G$  is co-countable,  $G/H$  is a countable group so there exists a free  $S_2$ -embedding  $\sigma : \omega^2 \rightarrow G/H$  such that  $\sigma = p \circ \sigma'$  where  $\sigma' : \omega^2 \rightarrow G$  is a free  $S_2$ -embedding in  $\tau$ . Since  $\mathcal{U}$  forms a base of open neighborhoods of  $0 \in G$  in  $\tau$  one can find  $\alpha \in \omega_1$  such that  $H_\beta \subseteq H$  for any  $\beta \geq \alpha$  and thus  $p_\beta \circ \sigma'$  is a free  $S_2$ -embedding. If  $\beta$  is large enough  $\sigma' = \sigma_\beta$  and thus by (f)  $p_\beta \circ \sigma_\beta$  is not a free  $S_2$ -embedding contradicting the choice of  $\sigma'$ .

**Remark 3.** It is an easy observation that by replacing (e) above with

(e') if  $V = \cup \{ B^2(\beta) : \beta \in A_\alpha \}$  has a nonempty interior in  $\tau_\alpha^2$  and  $(0, 0) \in \text{Int}(V)$  then there is  $U \in \mathcal{U}_\alpha$  such that  $(0, 0) \in U^2 \subseteq \text{Int}(V)$ .

and adjusting the definition of a saturated ordinal to

- (1) if  $\beta_1, \dots, \beta_n \in O_\alpha$  and  $K \subseteq G^2$  is compact (in  $\tau_\alpha^2$ ) then there exist  $\beta^1, \dots, \beta^k \in O_\alpha$  such that each  $B^2(\beta^i) \cap K$  is relatively open in  $K$ , and  $\overline{\cup_{i \leq n} B^2(\beta_i)} \subseteq \cup_{j \leq k} B^2(\beta^j) \subseteq \overline{\cup_{j \leq k} B^2(\beta^j)} \subseteq G^2 \setminus F$ .
- (2) for each  $\beta \in O_\alpha$   $t(\beta) < \alpha$  and there exists  $\beta \in O_\alpha$  such that  $0 \in B(\beta)$ .

after assuming  $F \subseteq G^2$  and defining  $O_\alpha$  and  $t(\beta)$  appropriately, one can construct a group  $G$  as above with the additional property that  $G^2$  is sequential (indeed, all finite powers of  $G$  can be made sequential after a minor change to the method above). By ‘trapping’ closed copies of  $S(\omega)$  in each  $G_\alpha$  using Lemma 2 (or noting that such a copy must exist in any sequential group that contains  $G_\alpha$ ) and using Lemmas 3 and 9 one can construct a  $G$  as above with the additional property that the only sequential subgroups of  $G$  are closed and uncountable (in fact, the  $G$  constructed above has this property automatically, as noted above). This, in turn, implies that every sequential subgroup of  $G$  is either countable and discrete or contains a compact subspace of uncountable pseudocharacter.

Finally, a more precise statement of Lemma 10 (with a modified proof) would allow a construction of such  $G$  with  $\text{so}(G) = \omega + 1$ .

The results above leave open a number of interesting questions.

**Question 1.** *Does there exist (consistently or in ZFC) a sequential group  $G$  such that all countable sequential subgroups of  $G$  are finite and all the quotients of  $G$  are either first countable or have uncountable pseudocharacter?*

Note that the construction of Example 2 produces a closed copy of  $S(\omega)$  in  $G$ . If  $G \times G$  is also sequential then no nontrivial quotient of  $G$  can be first countable. Indeed, otherwise some open homeomorphic image of  $G \times G$  would contain closed copies of both  $S(\omega)$  and  $D_\omega$  contradicting Lemma 3.

A very strong version of the question above is also open. Note that it becomes meaningful only when the negation of CH is assumed.



**Question 2.** *Does there exist a sequential group  $G$  such that every sequential subgroup of  $G$  is either finite or closed, uncountable, and of countable index and every quotient of  $G$  is either first countable or has a pseudocharacter  $> \omega_1$ ? Can such  $G$  (if exists) have any sequential order?*

Finally, if the quotient requirements are dropped, can the group be made  $k_\omega$ ?

**Question 3.** *Does there exist a sequential  $k_\omega$  group  $G$  such that every countable sequential subgroup of  $G$  is discrete (finite)? In particular does there exist a sequence as described in Example 1 that makes  $\mathbb{R}$  into such a group? Countably many sequences?*

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